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Integrable lattice gauge model based on soliton theory

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Abstract. A gauge model Lagrangian on a three-dimensional lattice is proposed. Under certain coinditions equations of motion are reduced to the Bäcklund transformation of Hirota's bilinear difference equation, which is solved by every solution ot the Kadomtsev-Petviashvili (KP) hierarchy. Generation of soliton solutions given in the Casorati determinant form is discussed explicitly.

1. Introduction

Lattice models have been studied in various fields of physics. Lattice gauge theory provides a useful tool for investigating the non-perturbative nature of particle physics at short distances [1]. After the classic paper by Onsager [2], solvable lattice models have been worked intensively in the field of statistical mechanics [3]. Moreover, recent studies of the Yang-Baxter relation revealed a close relation between various solvable statistical models on a two-dimensional lattice and knot theory [4] as well as conformal field theory [5]. Integrable lattice models also appear in soliton physics, known as the Toda lattice or its generalisations [6].

We have investigated [7], in our previous papers, Hirota's bilinear *difference* equation (HBDE) [8]. It is a nonlinear three-dimensional lattice equation which is completely integrable. We emphasise here that this equation can describe physical models in real three-dimensional lattice space in the sense that the three independent variables appear symmetrically. Moreover, it was pointed out [9] that this equation is satisfied by string amplitudes which determine the behaviour of elementary particles beyond the Planck scale.

The purpose of our present paper is to derive a Lagrangian associated with this equation, in order to clarify the physical background on which the equation is based. Instead of studying HBDE directly, however, we consider a linearised version of this equation by introducing gauge potentials [7]. Then it turns out that the corresponding Lagrangian is described by a lattice gauge model with a particular form of links. Under appropriate conditions the system appears symmetric under the exchange of fields defined on the lattice sites and the gauge potentials defined along the links. This remarkable feature, called duality, enables us to investigate the behaviour of solutions and shows the correlation between the fields and the gauge potentials. In section 3 we will discuss this feature in some detail for soliton solutions expressed in the form of the Casorati determinant.

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2. Lagrangian

First we review HBDE briefly. It is given by

$$\alpha f(\lambda+1,\mu,\nu)f(\lambda-1,\mu\,\nu)+\beta f(\lambda,\mu+1,\nu)f(\lambda,\mu-1,\nu) + \gamma f(\lambda,\mu,\nu+1)f(\lambda,\mu\,\nu-1)=0 \tag{1}$$

where α, β and γ are constants. As was discussed in [7], this equation can be derived as a consistency condition to the following linear problem:

$$\nabla_\pm \phi(l,m,n)=c_\pm E_\pm \phi(l,m,n) \tag{2}$$

where

$$\begin{aligned}\nabla_\pm &= U_\pm(l,m,n)(\mathrm{e}^{\partial_\pm}-1)U_\pm^{-1}(l,m,n)\\ E_\pm &= U_\mp(l,m,n)\,\mathrm{e}^{\partial_\pm \pm\partial_n}U_\mp^{-1}(l,m,n)\end{aligned} \tag{3}$$

with

$$\partial_+=\partial/\partial l \qquad \partial_-=\partial/\partial m \qquad \partial_n=\partial/\partial n$$

$$l=\frac{\lambda+\mu+\nu}{2} \qquad m=\frac{\mu-\lambda-\nu}{2} \qquad n=\nu$$

and $c_\pm$ are constants related to α and γ by $c_+c_-=-\gamma/\alpha$. In fact under the gauge condition

$$U_+(l,m,n)=U_-(l,m,n+1)\equiv U(l,m,n) \tag{4}$$

the coupled equation (2) is compatible if $U(l,m,n)$ satisfies (1). If U satisfies HBDE, ϕ is obtained by solving the second order *linear* difference equation

$$\nabla_-\nabla_+\phi(l,m,n)=c_+c_-E_+E_-\phi(l,m,n). \tag{5}$$

The remarkable feature of (2) is that it can be also rewritten as

$$\tilde\nabla_\pm\tilde\phi=c_\pm\tilde E_\pm\tilde\phi \tag{6}$$

where

$$\tilde\nabla_\pm\equiv\tilde U_\pm(1-\mathrm{e}^{-\partial_\pm})\tilde U_\pm^{-1} \qquad \tilde E_\pm\equiv-\tilde U_\mp\,\mathrm{e}^{-\partial_\pm\mp\partial_n}\tilde U_\mp^{-1} \tag{7}$$

and

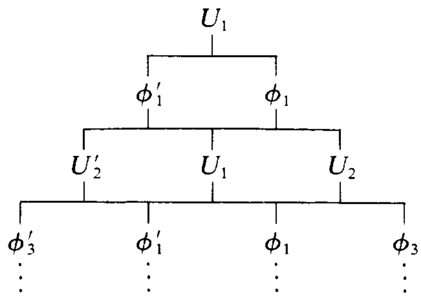
$$\tilde\phi(l,m,n)=U(l,m,n) \qquad \tilde U(l,m,n)=\phi(l,m,n) \tag{8}$$

under the gauge condition

$$\tilde U_-(l,m,n-1)=\tilde U_+(l,m,n)\equiv\tilde U(l,m,n). \tag{9}$$

In (6) the roles of the gauge potential U and the wavefunction ϕ in (2) are exchanged. We called [7] this property the 'duality' between U and ϕ. It is easy to prove that the compatibility condition of (6) again requires $\tilde U(l,m,n)$, hence $\phi(l,m,n)$ itself, to satisfy HBDE. Hence $\phi(l,m,n)$ solves HBDE if $U(l,m,n)$ does, and vice versa.

Suppose $U=U_1$ is a solution of HBDE. Solving (5) with this gauge potential U_1, we find two independent solutions ϕ_1 and ϕ_1'. If we use $\phi_1=\tilde U_1$ as a gauge potential of (6) we again obtain two independent solutions, say $\tilde\phi_2\equiv U_2$ and $\tilde\phi_2'\equiv U_2''$. Since (2) and (6) are identical, one of these solutions, say U_2'', must be U_1. Similarly if we use ϕ_1' as a potential of (6) we obtain another pair of solutions, one of which is equal to U_1. In this way (2) (or (6)) generates the auto-Bäcklund transformation as shown in the following diagram:



Our problem in this paper is to derive the Lagrangian which leads to the equation of motion given by (2) and (6). For this purpose let us see these equations in detail. First, we notice that there are four equations which are linear in $\phi(l, m, n)$ and $\tilde{\phi}(l, m, n)$. Second, they are covariant under the local gauge transformation:

$$U_{\pm}(l, m, n) \to V(l, m, n) U_{\pm}(l, m, n) \qquad \phi(l, m, n) \to V(l, m, n)\phi(l, m, n). \tag{10}$$

Therefore we consider a Lagrangian which contains four independent fields in bilinear form and is local gauge invariant.

There are two alternative cases of the choice of time variable. One is to regard one of three variables as a discrete time variable. In fact, originally [8], HBDE was first derived as a discrete analogue of the two-dimensional Toda lattice. Another is to introduce the time variable t besides l, m and n. The most general form of the Lagrangian of this type is given by

$$L = 2\mathrm{i} \sum_j \sum_{\alpha=1}^{4} \phi_j^{\alpha +}(\partial_t - \mathrm{i}A_{0,j})\phi_j^{\alpha} + h \sum_{j,k} \sum_{\alpha,\beta=1}^{4} \phi_j^{\alpha +} W_{j,k}^{\alpha,\beta} \phi_k^{\beta} + L_U \tag{11}$$

from which the equations of motion for the field ϕ_j^{α} are simply derived as

$$2\mathrm{i}(\partial_t - \mathrm{i}A_{0,j})\phi_j^{\alpha} + h \sum_k \sum_{\beta=1}^{4} W_{jk}^{\alpha\beta} \phi_k^{\beta} = 0. \tag{12}$$

In (11) L_U represents the kinetic term of the gauge fields. Since (2) and (6) are two pairs of separate equations we label α by two suffixes (σ, χ) with $\sigma = 1$, 2 and $\chi = \mathrm{R}$, L. comparing (2) and (6) with (12) we find

$$\begin{aligned}
&W_{l,m,n;l,m,n}^{\alpha\beta} = \delta_{\alpha\beta} \\
&W_{l,m,n;l\pm 1,m,n}^{\sigma\chi,\sigma'\chi} = -U_{\pm}(l, m, n)\, \mathrm{e}^{\pm\partial_+} U_{\pm}^{-1}(l, m, n)\delta_{\sigma 1}\delta_{\sigma' 2}\delta_{\chi\pm} \\
&W_{l,m,n;l,m\pm 1,n}^{\sigma\chi,\sigma'\chi} = -U_{\mp}(l, m, n)\, \mathrm{e}^{\pm\partial_-} U_{\mp}^{-1}(l, m, n)\delta_{\sigma 2}\delta_{\sigma' 1}\delta_{\chi\pm} \\
&W_{l,m,n;l\pm 1,m,n\pm 1}^{\sigma\chi,\sigma'\chi} = c_+ U_{\mp}(l, m, n)\, \mathrm{e}^{\pm\partial_+ \pm\partial_n} U_{\mp}^{-1}(l, m, n)\delta_{\sigma 1}\delta_{\sigma' 2}\delta_{\chi\pm} \\
&W_{l,m,n;l,m\pm 1,n\mp 1}^{\sigma\chi,\sigma'\chi} = c_- U_{\pm}(l, m, n)\, \mathrm{e}^{\pm\partial_- \mp\partial_n} U_{\pm}^{-1}(l, m, n)\delta_{\sigma 2}\delta_{\sigma' 1}\delta_{\chi\pm} \\
&W_{jk}^{\alpha\beta} = 0 \qquad \text{otherwise.}
\end{aligned} \tag{13}$$

Note that $\chi = \mathrm{R}$, L, which are abbreviated as $\chi = -, +$ in (13), correspond to the right- and left-moving components, respectively.

We will not consider the kinetic term L_U of the gauge fields in this paper, but simply ignore it so that the gauge potentials $U_\pm$ are not dynamical fields. The Lagrangian is still invariant under the local guage transformation of the type (10). We can fix $U_\pm$ either by hand or by solving them as functions of the ϕ. Instead, we can also fix $U_\pm$ such that (12) becomes compatible when we require

$$\phi_j^{1\chi} = \phi_j^{2\chi} \qquad \chi = \mathrm{R}, \mathrm{L}. \tag{14}$$

Under this constraint (12) becomes (2) and (6) with $\phi_j^{1\mathrm{L}} = \phi_j^{2\mathrm{L}} = \phi_j$ and $\phi_j^{1\mathrm{R}} = \phi_j^{2\mathrm{R}} = \tilde{\phi}_j$, if the t dependence is ignored. Similarly we derive the same set of equations for $\phi_j^{\alpha\dagger}$ as those for ϕ_j^α under the same conditions. Therefore we conclude that the system described by the Lagrangian (11) is characterised by HBDE when the gauge potential U satisfies HBDE and if there is no t dependence.

The system described by the Lagrangian (11) is a lattice gauge model in which fields ϕ_j defined on the lattice site j interact with each other through the gauge potential defined along the links. It is a three-dimensional lattice which consists of layers of two-dimensional square lattice. Two adjacent layers couple each other with strength $c_\pm$ through particular links. If we do not consider the variable t, the Lagrangian describes a two-dimensional lattice which evolves along the discrete time variable n.

3. Behaviour of solutions

3.1. *Solutions to the* KP *hierarchy*

The general solution to the KP hierarchy is known. It was shown by Miwa [10] that every solution to the KP hierarchy also satisfies HBDE. In particular, the quasi-periodic solution to HBDE has been given explicitly as [9]

$$\tau(p, q, r) = \prod_{i,j=0}^{\infty} \left(\frac{E(z_i, z_j)}{z_i - z_j} \right)^{(1/2)p_i p_j} \theta\left(\zeta + \sum_{j=0}^{\infty} p_j \int^{z_j} \omega \right) \tag{15}$$

where θ and E are the Riemann theta function and the prime form, whereas ω and ζ are the first kind of Abel differential and some constants. The variables p, q, r are any three out of $\{p_j; j = 0, 1, 2, 3, \ldots, \infty\}$ and are related to l, m, n and hence λ, μ, ν of (1), by

$$l = p + q + r + \tfrac{3}{2} \qquad m = -q - \tfrac{1}{2} \qquad n = p + q + 1. \tag{16}$$

If we substitute (15) into HBDE we obtain Fay's trisecant formula [11], which characterises the algebraic curves, along with

$$-\frac{\beta}{\alpha} = \frac{z_q(z_r - z_p)}{z_p(z_r - z_q)} \qquad c_+ c_- = -\frac{\gamma}{\alpha} = \frac{z_r(z_q - z_p)}{z_p(z_q - z_r)}. \tag{17}$$

Another important solution is the N-soliton solution. An explicit form of the N-soliton solution to the KP hierarchy has been given by using a Wronskian [12]:

$$W[\varphi^{(0)}, \varphi^{(1)}, \ldots, \varphi^{(N-1)}] \equiv \begin{vmatrix} \varphi_1^{(0)} & \varphi_1^{(1)} & \cdots & \varphi_1^{(N-1)} \\ \varphi_2^{(0)} & \varphi_2^{(1)} & \cdots & \varphi_2^{(N-1)} \\ \vdots & \vdots & \cdots & \vdots \\ \varphi_N^{(0)} & \varphi_N^{(1)} & \cdots & \varphi_N^{(N-1)} \end{vmatrix} \tag{18}$$

where

$$\varphi_i^{(n)} \equiv \frac{\partial^n}{\partial t_1^n} \varphi_i \qquad i=1,2,3,\ldots,N \tag{19}$$

$$\varphi_i = \exp\left(\sum_{n=1}^{\infty} t_n a_i^n\right) + \exp\left(\sum_{n=1}^{\infty} t_n b_i^n\right) \tag{20}$$

and $\varphi^{(n)}$ are column vectors. The variables t_n of the KP hierarchy can be transformed into those of HBDE by [10]

$$t_n = \frac{1}{n}\sum_{j=0}^{\infty} p_j z_j^n \qquad n=1,2,3,\ldots. \tag{21}$$

Accordingly, $\varphi_i^{(n)}$ are transformed into

$$\varphi_i^{(n)} = a_i^n \prod_{j=0}^{\infty} (1-a_i z_j)^{-p_j} + b_i^n \prod_{j=0}^{\infty} (1-b_i z_j)^{-p_j} \qquad \varphi_i = \varphi_i^{(0)}. \tag{22}$$

3.2. *Soliton solutions in Casorati determinant form*

Denote one of the p_j as s and write the corresponding φ_i as $\varphi_i(s)$. Then the following relation is true:

$$\varphi_i^{(n+1)}(s) = \frac{1}{z_s}(\varphi_i^{(n)}(s) - \varphi_i^{(n)}(s-1)) \qquad s \in \{p_j\}. \tag{23}$$

If we apply this to $\varphi^{(N-1)}, \varphi^{(N-2)}, \ldots$ of (18) successively we obtain

$$\begin{aligned}
&W[\varphi^{(0)}(s), \varphi^{(1)}(s), \ldots, \varphi^{(N-1)}(s)] \\
&\quad = -\frac{1}{z_s} W[\varphi^{(0)}(s), \varphi^{(1)}(s), \ldots, \varphi^{(N-2)}(s), \varphi^{(N-2)}(s-1)] \\
&\quad = \frac{1}{z_s^2} W[\varphi^{(0)}(s), \varphi^{(1)}(s), \ldots, \varphi^{(N-3)}(s), \varphi^{(N-3)}(s-1), \varphi^{(N-2)}(s-1)] \\
&\quad \vdots \\
&\quad = \left(\frac{-1}{z_s}\right)^{N-1} W[\varphi^{(0)}(s), \varphi^{(0)}(s-1), \varphi^{(1)}(s-1), \ldots, \varphi^{(N-2)}(s-1)].
\end{aligned}$$

We can continue this procedure until we get

$$\left(\frac{-1}{z_s}\right)^{N(N-1)/2} W[\varphi(s), \varphi(s-1), \ldots, \varphi(s-N+1)].$$

The determinant we have just obtained has the form of a Casorati determinant [13], a discrete version of the Wronskian. Since an overall constant is irrelevant in the bilinear form of equations, we define the N-soliton solution in the Casorati determinant form by†

$$\tau^N(p) = W[\varphi(s+1), \varphi(s+2), \ldots, \varphi(s+N)] \tag{24}$$

where we have replaced s by $s+N$ and rearranged the order of columns.

† We learned from Professor R Hirota that he also has derived the same type of expression, from a talk presented at the conference held at Res. Institute of Mathematical Sciences (Kyoto University) on 22 April 1988, and through private communication.

3.3. Generation of soliton solutions via duality

The Lagrangian given by (11) is a coupled system of the wavefunctions ϕ and the gauge potential U. We have seen in section 2 that they are related by duality. We would like to clarify this scheme through the examination of soliton solutions explicitly. For this purpose let us write down (2) explicitly:

$$U(l, m, n)\phi(l+1, m, n) - U(l+1, m, n)\phi(l, m, n) - c_+ U(l, m, n-1)\phi(l+1, m, n+1) = 0 \tag{25}$$

$$U(l, m, n-1)\phi(l, m+1, n) - U(l, m+1, n-1)\phi(l, m, n) - c_- U(l, m, n)\phi(l, m+1, n-1) = 0. \tag{26}$$

It is convenient to use the symmetric variables p_j instead of l, m, n, defined by (16). We are now going to show that (25) and (26) are satisfied by

$$U(l, m, n-1) = \left(\frac{z_p - z_r}{z_p}\right)^l \left(\frac{z_p - z_r}{z_q - z_r}\right)^m \tau^{N+1}(p, q, r+1) \qquad \phi(l, m, n) = \tau^N(p, q, r) \tag{27}$$

where $\tau^N(p, q, r)$ is the N-soliton solution of (24) and (z_p, z_q, z_r) are the local coordinates associated with p, q, r, respectively, which appear in the expression of φ in (22). In this proof the following identities satisfied by φ, which hold for arbitrary three variables u, v, w in $\{p_j\}$, are useful:

$$(z_u - z_v)\varphi(u+1, v+1) = z_u\varphi(u+1, v) - z_v\varphi(u, v+1) \tag{28}$$

$$(z_u - z_v)\varphi(u+1, v+1, w) + (z_v - z_w)\varphi(u, v+1, w+1) + (z_w - z_u)\varphi(u+1, v, w+1) = 0. \tag{29}$$

The substitution of (27) into the left-hand sides of (25) and (26) yields

$$\text{LHS of (25)} \propto \tau^{N+1}(p+1, q, r)\tau^N(p, q, r+1) - \frac{z_p - z_r}{z_p}\tau^{N+1}(p+1, q, r+1)\tau^N(p, q, r) - c_+\tau^{N+1}(p, q, r+1)\tau^N(p+1, q, r) \tag{30}$$

$$\text{LHS of (26)} \propto \tau^{N+1}(p, q+1, r+1)\tau^N(p+1, q, r) - \frac{z_p - z_r}{z_q - z_r}\tau^{N+1}(p+1, q, r+1)\tau^N(p, q+1, r) - c_-\tau^{N+1}(p+1, q+1, r)\tau^N(p, q, r+1). \tag{31}$$

Let us consider (31) first. We want to know the difference of each factor in (31) from the expression of $\tau^N(p, q, r)$ given by the Casorati determinant (24). As an example, we look at $\tau^{N+1}(p, q+1, r+1)$. According to (24) it is given by

$$\tau^{N+1}(p, q+1, r+1) = W[\hat{\varphi}(p, q+1, r+1, s), \hat{\varphi}(p, q+1, r+1, s+1), \ldots, \hat{\varphi}(p, q+1, r+1, s+N)] \tag{32}$$

where $\hat{\varphi}$ denotes the $(N+1)$-component vector associated with the $(N+1)$-solitons. Applying the identity (28) repeatedly, we see that this is equal to

$$\left(\frac{z_s^2}{(z_q-z_s)(z_r-z_s)}\right)^{N-1} W[\hat{\varphi}(p, q+1, r+1, s), \hat{\varphi}(p, q+1, r+1, s+1), \hat{M}].$$

Here $\hat{M}$ is the $(N+1)\times(N-1)$ matrix of the form

$$\hat{M}=(\hat{\varphi}(p, q, r, s+2), \hat{\varphi}(p, q, r, s+3), \ldots, \hat{\varphi}(p, q, r, s+N)).$$

A similar argument will show that factors produced by this procedure in front of each term of (31) are the same, and hence can be factored out from the expression. Thus we see that the right-hand side of (31) is proportional to

$$\begin{aligned}
W[\hat{\varphi}(p, q+1, r+1, s), &\hat{\varphi}(p, q+1, r+1, s+1), \hat{M}]\, W[\varphi(p+1, q, r, s+1), M] \\
&-\frac{z_p-z_r}{z_q-z_r} W[\hat{\varphi}(p+1, q, r+1, s), \hat{\varphi}(p+1, q, r+1, s+1), \hat{M}] \\
&\times W[\varphi(p, q+1, r, s+1), M] \\
&-c_- W[\hat{\varphi}(p+1, q+1, r, s), \hat{\varphi}(p+1, q+1, r, s+1), \hat{M}] \\
&\times W[\varphi(p, q, r+1, s+1), M]. \qquad (33)
\end{aligned}$$

In the above expression M is the $N\times(N-1)$ matrix defined by

$$M=(\varphi(p, q, r, s+2), \varphi(p, q, r, s+3), \ldots, \varphi(p, q, r, s+N)).$$

We will show in the appendix that (33) turns out to be

$$\begin{aligned}
W[\hat{\varphi}(p+1, q+1, r, s), &\hat{\varphi}(p, q, r+1, s+1), \hat{M}]\, W[\varphi(p, q+1, r, s+1), M] \\
&- W[\hat{\varphi}(p, q+1, r, s+1), \hat{\varphi}(p, q, r+1, s+1), \hat{M}] \\
&\times W[\varphi(p+1, q+1, r, s), M] \\
&- W[\hat{\varphi}(p+1, q+1, r, s), \hat{\varphi}(p, q+1, r, s+1), \hat{M}] \\
&\times W[\varphi(p, q, r+1, s+1), M] \qquad (34)
\end{aligned}$$

provided $c_-=(z_q-z_p)/(z_q-z_r)$. A detailed inspection will reveal that the first two terms of (34) are exactly the Laplace expansion along the N-rows in the bottom of the determinant of the following $(2N+1)\times(2N+1)$ matrix:

$$\begin{pmatrix}
\varphi(p, q, r+1, s+1) & M & \varphi(p+1, q+1, r, s) & \varphi(p, q+1, r, s+1) & M \\
\varphi'(p, q, r+1, s+1) & M' & \varphi'(p+1, q+1, r, s) & \varphi'(p, q+1, r, s+1) & M' \\
0 & 0 & \varphi(p+1, q+1, r, s) & \varphi(p, q+1, r, s+1) & M
\end{pmatrix} \qquad (35)$$

whereas the last term with the negative sign is the one along the N columns in the left of the same matrix, where φ' and M' mean the $(N+1)$th components of $\hat{\varphi}$ and $\hat{M}$, respectively. In other words, the sum of the first two terms is equal to the negative of the last term. Therefore (34) vanishes. This ensures that (26) is satisfied by the soliton solutions of the Casorati determinant form when $c_-=(z_q-z_p)/(z_q-z_r)$.

Expression (25) will be shown to vanish mostly in parallel to the above reasoning. In this case the corresponding matrix is

$$\begin{pmatrix} \varphi(p,q,r+1,s+1) & M & \varphi(p+1,q,r,s) & \varphi(p,q,r,s+1) & M \\ \varphi'(p,q,r+1,s+1) & M' & \varphi'(p+1,q,r,s) & \varphi'(p,q,r,s+1) & M' \\ 0 & 0 & \varphi(p+1,q,r,s) & \varphi(p,q,r,s+1) & M \end{pmatrix}$$

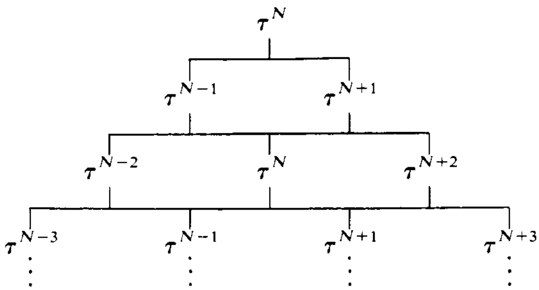
if c_+ is given by $c_+ = z_r/z_p$. We have again obtained

$$c_+c_- = \frac{z_r(z_q - z_p)}{z_p(z_q - z_r)} \tag{36}$$

the same expression as (17). It is interesting that the combination of the coupling constants c_+c_- is given by the same cross ratio (36) of the parameters appearing in the Miwa transformation (21) irrespective to the explicit form of solutions, either the quasi-periodic or N-soliton solutions.

3.4. *Bäcklund transformation*

From the above consideration we convinced ourselves that (2) generates a N-soliton solution from a $(N+1)$-soliton solution. This is a generalisation of the result shown in the case of the two-dimensional Toda lattice [14]. Equation (2) can be also expressed as its dual equation (6), which enables us to deduce that (6) generates the $(N+1)$-soliton solution from the N-soliton solution. Furthermore, from the discussion of the end of section 2, we can see that another solution of the linear equation (5) must be a $(N+2)$-soliton solution when the potential U is the $(N+1)$-soliton. Hence, corresponding to the diagram for the general case of the transformation, we obtain the net diagram of the Bäcklund transformation associated with the soliton solution:



4. Discussion

We have derived a lattice model Lagrangian (11) in three dimensions which is gauge symmetric and is integrable under appropriate conditions. This model describes two-dimensional square lattices which also couple with each other along their normal direction through the interaction characterised by the two coupling constants $c_\pm$. The equation of motion (2) for the fields ϕ possesses dual symmetry, i.e. they are symmetric under the exchange of the roles played by the matter fields and the gauge potential fields. In particular this remarkable property of the equations proves that all of the fields ϕ, $\tilde{\phi}$ and U are characterised by a single nonlinear equation, i.e. the HBDE, when the condition (14) is fulfilled. Equations (2) and (6) themselves provide a scheme of an auto-Bäcklund transformation and enables us to solve HBDE successively.

The HBDE is solved by every solution of the KP hierarchy, an infinite series of integrable soliton equations. There exists a transformation, i.e. the Miwa transformation (21), which maps solutions of the KP hierarchy to those of HBDE. Employing this transformation, we constructed explicitly the quasi-periodic solutions and the N-soliton solutions which satisfy HBDE. The Bäcklund transformation was confirmed explicitly in the case of soliton solutions and we have found that (2) and (6) generate $(N-1)$- and $(N+1)$-soliton solutions, respectively, from the N-soliton solution. This is a particular example of the fact known in the theory of the KP hierarchy [10] that an addition of a soliton to a τ function generates a Bäcklund transformation. In the quasi-periodic solution this amounts to subtracting or adding one of the p_j to the τ function given by (15).

To conclude our discussion let us see some properties of the system described by the Lagrangian (11). First of all, this Lagrangian possesses the gauge symmetry represented by (10). The origin of this symmetry lies on the fact that the field ϕ always appears together with the gauge potential U in the Lagrangian, as we can readily see by the substitution of the expression for the link operators (13) into the Lagrangian. The dual symmetry between ϕ and U in (2) or (6) follows again to this fact provided ϕ satisfies (14), U satisfies (4) and the time dependence is irrelevant.

This symmetry induces a strong correlation between these two fields. In the case of soliton solutions the N-soliton potential generates $N\pm 1$ solitons. The change of the fields by either subtaction or addition of one siliton will not cause a big difference of behaviour between the gauge potential and matter field in as much as the system contains many solitons. It will be quite interesting if one could find lattice systems described by such a Lagrangian.

Appendix

In this appendix we prove (34) starting from (33);

$$\begin{aligned}
&W[\hat{\varphi}(p,q+1,r+1,s),\hat{\varphi}(p,q+1,r+1,s+1),\hat{M}]\,W[\underline{\varphi(p+1,q,r,s+1)}_{29},M]\\
&\qquad-\frac{z_p-z_r}{z_q-z_r}W[\hat{\varphi}(p+1,q,r+1,s+1),\underline{\hat{\varphi}(p+1,q,r+1,s+1)}_{28},\hat{M}]\\
&\qquad\times W[\varphi(p,q+1,r,s+1),M]\\
&\qquad-c_-W[\hat{\varphi}(p+1,q+1,r,s),\underline{\hat{\varphi}(p+1,q+1,r,s+1)}_{28},\hat{M}]\\
&\qquad\times W[\varphi(p,q,r+1,s+1),M].
\end{aligned}\tag{33}$$

First apply (28) and (29) to the terms indicated by the underlines with corresponding numbers, to get

$$\begin{aligned}
&W[\hat{\varphi}(p,q+1,r+1,s),\hat{\varphi}(p,q+1,r+1,s+1),\hat{M}]\\
&\qquad\times W\left[\frac{z_p-z_q}{z_p-z_s}\varphi(p+1,q+1,r,s)+\frac{z_q-z_s}{z_p-z_s}\varphi(p,q+1,r,s+1),M\right]\\
&\qquad+\frac{z_p-z_r}{z_q-z_r}\,\frac{z_s}{z_p-z_s}\,W[\underline{\hat{\varphi}(p+1,q,r+1,s)}_{29},\hat{\varphi}(p,q,r+1,s+1),\hat{M}]\\
&\qquad\times W[\varphi(p,q+1,r,s+1),M]\\
&\qquad+c_-\frac{z_s}{z_p-z_s}W[\hat{\varphi}(p+1,q+1,r,s),\hat{\varphi}(p,q+1,r,s+1),\hat{M}]\\
&\qquad\times W[\varphi(p,q,r+1,s+1),M].
\end{aligned}$$

The first two terms of this expression can be further rewritten as

$$\begin{aligned}
&\frac{z_p - z_q}{z_p - z_s} W[\underline{\hat{\varphi}(p, q+1, r+1, s)}_{29}, \underline{\hat{\varphi}(p, q+1, r+1, s+1)}_{28}, \hat{M}] \\
&\qquad \times W[\varphi(p+1, q+1, r, s), M] \\
&\qquad + \frac{z_q - z_s}{z_p - z_s} W[\hat{\varphi}(p, q+1, r+1, s), \underline{\hat{\varphi}(p, q+1, r+1, s+1)}_{28}, \hat{M}] \\
&\qquad \times W[\varphi(p, q+1, r, s+1), M] \\
&\qquad + \frac{z_p - z_r}{z_q - z_r} \frac{z_s}{z_p - z_s} \frac{z_p - z_q}{z_p - z_r} W[\hat{\varphi}(p+1, q+1, r, s), \hat{\varphi}(p, q, r+1, s+1), \hat{M}] \\
&\qquad \times W[\varphi(p, q+1, r, s+1), M] \\
&\qquad + \frac{z_p - z_r}{z_q - z_r} \frac{z_s}{z_p - z_s} \frac{z_q - z_r}{z_p - z_r} W[\hat{\varphi}(p, q+1, r+1, s), \hat{\varphi}(p, q, r+1, s+1), \hat{M}] \\
&\qquad \times W[\varphi(p, q+1, r, s+1), M].
\end{aligned}$$

If we apply (28) to the second term it cancels the last term. Also applying (28) and then (29) to the first term we obtain

$$\begin{aligned}
&= -\frac{z_s}{z_p - z_s} \frac{z_p - z_q}{z_q - z_r} (W[\hat{\varphi}(p, q+1, r, s+1), \hat{\varphi}(p, q, r+1, s+1), \hat{M}] \\
&\qquad \times W[\varphi(p+1, q+1, r, s), M] \\
&\qquad - W[\hat{\varphi}(p+1, q+1, r, s), \hat{\varphi}(p, q, r+1, s+1), \hat{M}] \\
&\qquad \times W[\varphi(p, q+1, r, s+1), M]).
\end{aligned}$$

Combining all together, we obtain (34), if c_- is given by $(z_q - z_p)/(z_q - z_r)$.

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